HW 3 Solutions

P1)

A.) $\frac{d}{dt}\mathcal{U}(t) = \int_U \partial u/\partial t\,(t,x)\,dx = \int_U \Delta u\,dx = \int_{\partial U} \partial u/\partial\eta\, dS = 0.$

B.) $\frac{d}{dt}\eta(t) = \int_U 2u\frac{\partial u}{\partial t}\,dx = \int_U 2u(\Delta u)\,dx.$ Let ν be the outward normal to ∂U at x, $\nu = \nu(x)$, and

$$\eta'(t) = \int_{\partial U} 2u\frac{\partial u}{\partial \nu}\,dS - \int_U |\nabla u|^2 dx = -\int_U |\nabla u|^2 dx \leq 0.$$

C.) If u_1, u_2 are two such solutions, $w = u_1 - u_2$

has ~~$w(0,x) = 0$~~ ~~$= \frac{\partial w}{\partial \nu}$~~

$$w|_{\partial U} = 0 = \frac{\partial w}{\partial \nu}$$

By (B), $\eta_w(0) = 0$ and $0 \leq \eta_w(t) \leq \eta_w(0)$ giving $\eta_w(t) = 0$ for all time. Then, $w(t,x) = 0$ for all x, t.

P2)

A.) $m(I_0) = 1.$

B.) $m(I_1) = 2\cdot 1/3$

$m(I_2) = 4\cdot 1/9$

$m(I_k) = (2/3)^k$

C.) $m(C) \leq (2/3)^k$ for all $k \Rightarrow m(C) = 0.$

P.3) ~> Typo, f_n was supposed to be $e^{-n^2 x}$

A.) $\|f_n\|_1 = \int_0^\infty n e^{-n^2 x} dx = 1/n$

$\|f_n\|_1 \to 0$ as $n \to \infty$

$\|f_n\|_2 = \int_0^\infty n^2 e^{-2n^2 x} dx = 1/2$

B.) $\|g_n\|_1 = 1$

$\|g_n\|_2 = 1/n \quad \to 0$ as $n \to \infty$

C.) $\|f_n\|_\infty = n$

$\|g_n\|_\infty = 1/n \quad \to 0$ as $n \to \infty$

P4.)

A.) $\eta'(t) = \int_u 2u \, \partial u/\partial t \, dx = \langle 2u, \partial u/\partial t \rangle \leq \|2u\|_{L^2} \|\partial u/\partial t\|_{L^2}$

$\leq 4\eta(t) \int_\Omega |\partial u/\partial t|^2 dt$

B.) $\eta''(t) = \int_u 2(\partial u/\partial t)(\partial u/\partial t) dx + \underbrace{\int_u 2u \Delta(\partial u/\partial t) dx}_{I}$

$I = \int_{\partial u} 2u \frac{\partial}{\partial t}(\partial u/\partial \eta) dx - \int_{\partial u} 2 \, \partial u/\partial \eta \, \partial u/\partial t \, dx + \int_u 2 \Delta u \frac{\partial u}{\partial t} dx$

$= \int_u 2(\partial u/\partial t)^2 dx$

or $\eta''(t) = \int_u 4(\partial u/\partial t)^2 dx$

C.) $\eta(t) = 0$ for all time by Part B of P1, so $u(t,x) = 0$ for all time t.